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SHORTER COMMUNICATIONS

HEAT TRANSFER FROM A SPHERE TO AN INFINITE MEDIUM*

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NOMENCLATURE

k,	thermal conductivity [W/m K];
T,	temperature [K];
R,	sphere radius [m];
T_{if},	interfacial temperature [K];
F,	$= rT$;
$\tilde{A}(s)$,	Laplace transform of $A(t)$;
$q/4\pi R^2$,	heat flux, $= -k\left.\dfrac{\partial T}{\partial r}\right\|_{r=R}$ [W/m^2].

Greek symbols

α,	thermal diffusivity; $(k/\rho c_p)$ [m^2/s];
ρ,	fluid density [kg/m^3];
τ,	relaxation time R^2/α [s].

Subscripts

1,	related to sphere;
2,	related to medium.

1. HEAT-TRANSFER EQUATIONS AND BOUNDARY CONDITIONS

WE ARE interested in knowing the heat fluxes and temperature profiles for the conduction heat-transfer problem of having a sphere initially at temperature T_{1i} inside a medium initially at temperature T_{2i}. The heat-transfer equations governing this process are:

$$\frac{\partial T_1}{\partial t} = \alpha_1 \frac{1}{r}\frac{\partial^2}{\partial r^2}[rT_1(r,t)], \quad \frac{\partial T_2}{\partial t} = \alpha_2 \frac{1}{r}\frac{\partial^2}{\partial r^2}[rT_2(r,t)]. \tag{1}$$

Here the subscript one pertains to the sphere and two to the infinite medium. At the interface R one has

$$T_1 = T_2, \quad k_1\frac{\partial T_1}{\partial r} = k_2\frac{\partial T_2}{\partial r}. \tag{2}$$

The initial state of the system is that at $t = 0$, $T_1 = T_{1i}$, $T_2 = T_{2i}$. Introducing the variable F by the relation

$$F = r(T - T_i) \tag{3}$$

so that $T = T_i + F/r$, we find that F obeys the one-dimensional heat-transfer equation

$$\frac{\partial F_1}{\partial t} = \alpha_1\frac{\partial^2 F_1}{\partial r^2}, \quad \frac{\partial F_2}{\partial t} = \alpha_2\frac{\partial^2 F_2}{\partial r^2}, \tag{4}$$

as well as the initial condition that at $t = 0$, $F_1 = F_2 = 0$. Introducing the Laplace transform $\tilde{F}_1(r,s)$ via

$$\tilde{F}(r,s) = \int_0^\infty e^{-st}F(r,t)\,dt$$

one has that $F(r,s)$ satisfies in each substance

$$s\tilde{F} - \alpha\frac{\partial^2\tilde{F}}{\partial r^2} = 0. \tag{5}$$

The solution of equation (5) satisfying the boundary conditions is

$$\begin{aligned}\tilde{F}_1(rs) &= \frac{k_2(T_{2i}-T_{1i})R\sinh[(s/\alpha_1)^{1/2}r][(s\tau_2)^{1/2}+1]}{s[k_1(s\tau_1)^{1/2}\cosh(s\tau_1)^{1/2}+k_2(s\tau_2)^{1/2}\sinh(s\tau_1)^{1/2}+(k_2-k_1)\sinh(s\tau_1)^{1/2}]}\\ \tilde{F}_2(r,s) &= \frac{k_1R(T_{1i}-T_{2i})\exp[-(s/\alpha_2)^{1/2}(r-R)][(s\tau_1)^{1/2}\cosh(s\tau_1)^{1/2}-\sinh(s\tau_1)^{1/2}]}{s\{k_1[(s\tau_1)^{1/2}\cosh(s\tau_1)^{1/2}-\sinh(s\tau_1)^{1/2}]+k_2\sinh(s\tau_1)^{1/2}[(s\tau_2)^{1/2}+1]\}}\end{aligned} \tag{6}$$

where $\tau_i = R^2/\alpha_i$.

At large s one has

$$\begin{aligned}\tilde{F}_1(rs) &\simeq \frac{k_2R(T_{2i}-T_{1i})}{s\Sigma}\left[(\tau_2)^{1/2}+\frac{k_1[(\tau_1)^{1/2}+(\tau_2)^{1/2}]}{\Sigma(s)^{1/2}+(k_2-k_1)}\right]\{\exp[-(s/\alpha_1)^{1/2}(R-r)]-\exp[-(s/\alpha_1)^{1/2}(R+r)]\}\\ \tilde{F}_2(rs) &\simeq \frac{k_1R(T_{1i}-T_{2i})}{s\Sigma}\exp[-(s/\alpha_2)^{1/2}(r-R)]\left[(\tau_1)^{1/2}-\frac{k_2[(\tau_1)^{1/2}+(\tau_2)^{1/2}]}{\Sigma(s)^{1/2}+(k_2-k_1)}\right]\\ \Sigma &= k_1(\tau_1)^{1/2}+k_2(\tau_2)^{1/2}.\end{aligned} \tag{6a}$$

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2. SPECIAL CASE OF IDENTICAL THERMAL PROPERTIES

When $k_1 = k_2$, $\alpha_1 = \alpha_2$ (that is the same liquid in 1 and 2), one gets a drastic simplification and has that equation (6) becomes

$$\tilde{F}_1 = \frac{(T_{2i}-T_{1i})}{2s} R[1+(s\tau)^{-1/2}]\{\exp[-(s/\alpha_1)^{1/2}(R-r)] - \exp[-(s/\alpha_1)^{1/2}(R+r)]\};$$

$$\tilde{F}_2 = \frac{R(T_{1i}-T_{2i})}{2s}\{\exp[-(s/\alpha_1)^{1/2}(r-R)] + \exp[-(s/\alpha_1)^{1/2}(R+r)]\}$$
$$+\frac{R(T_{1i}-T_{2i})}{2s(s\tau)^{1/2}}\{\exp[-(s/\alpha_1)^{1/2}(r+R)] - \exp[-(s/\alpha_1)^{1/2}(r-R)]\}. \quad (7)$$

Using standard tables of Laplace transforms we obtain for all r

$$T = T_{2i} - \frac{(T_{1i}-T_{2i})}{2}\left[\operatorname{erfc}\left(\frac{r+R}{2(\alpha t)^{1/2}}\right) - \operatorname{erfc}\left(\frac{r-R}{2(\alpha t)^{1/2}}\right)\right]$$
$$+\frac{R}{r}\frac{(T_{1i}-T_{2i})}{(\pi)^{1/2}}\left(\frac{t}{\tau}\right)^{1/2}\{\exp[-\tfrac{1}{4}(r+R)^2/(\alpha t)] - \exp[-\tfrac{1}{4}(r-R)^2/\alpha t]\} \quad (8)$$

where $\tau = R^2/\alpha$ and $\operatorname{erfc}(x) = 1-\operatorname{erf}(x)$ is the complementary error function. The interfacial heat flux is given by

$$\frac{q(t)}{4\pi R^2} = \frac{-k}{\pi^{1/2}R}(T_{2i}-T_{1i})\{(t/\tau)^{1/2}[\exp(-\tau/t)-1] + \tfrac{1}{2}(\tau/t)^{1/2}[1+\exp(-\tau/t)]\}. \quad (9)$$

3. GENERAL CASE

For the general case we can use the Laplace transform inversion formula to obtain a real integral representation for $F_1(r,t)$ and $F_2(r,t)$ by contour distortion. We have

$$F(r,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\tilde{F}(r,s)\,ds. \quad (10)$$

Since $\tilde{F}_1$ and $\tilde{F}_2$ have poles at $s = 0$ and a branch point at $s = 0$ we use the standard contour deformation to obtain

$$F(r,t) = Res\,\tilde{F}(r,s=0) + \frac{1}{2\pi i}\int_{-\infty,\,AB}^{0} e^{st}\tilde{F}(r,s)\,ds + \frac{1}{2\pi i}\int_{0,\,CD}^{-\infty} e^{st}\tilde{F}(r,s)\,ds. \quad (11)$$

Setting $s = \rho e^{i\theta}$, we have $\theta = \pi$ along CD and $\theta = -\pi$ along AB. Thus $(s)^{1/2} = i(\rho)^{1/2}$ on CD and $-i(\rho)^{1/2}$ on AB. We find $\int_{AB} = -(\int_{CD})^*$ so that

$$F(r,t) = Res\,\tilde{F}(r,s=0) + \frac{1}{\pi}Im\int_{0,\,CD}^{-\infty} e^{st}\tilde{F}(r,s)\,ds. \quad (12)$$

Letting $\rho = u^2\alpha_1/R^2$ and using $T = T_i + F/r$ we obtain

$$T_1 = T_{2i} + \frac{2}{\pi}k_1k_2(T_{2i}-T_{1i})(\alpha_1\alpha_2)^{1/2}\frac{R}{r}\int_0^\infty du\exp(-u^2t/\tau_1)(u\cos u - \sin u)\sin\left(\frac{ur}{R}\right)$$
$$\times[\alpha_1k_2^2u^2\sin^2 u + \alpha_2\{k_1(u\cos u - \sin u) + k_2\sin u\}^2]^{-1} \quad (13)$$

$$T_2 = T_{2i} + \frac{2}{\pi}k_1(\alpha_2)^{1/2}\frac{R}{r}(T_{2i}-T_{1i})\int_0^\infty \frac{du}{u}\exp(-u^2t/\tau_1)(u\cos u - \sin u)$$
$$\times\frac{k_2(\alpha_1)^{1/2}\cos\gamma u\sin u + (\alpha_2)^{1/2}\sin\gamma[k_1(u\cos u - \sin u) + k_2\sin u]}{\alpha_1k_2^2u^2\sin^2 u + \alpha_2[k_1(u\cos u - \sin u) + k_2\sin u]^2} \quad (14)$$

where $\gamma = (\alpha_1/\alpha_2)^{1/2}u[(r-R)/r]$, $\tau_1 = R^2/\alpha_1$, $\tau_2 = R^2/\alpha_2$.

Thus we have

$$\frac{q(t)}{4\pi R^2} = -\frac{2}{\pi}k_1^2k_2\frac{(T_{2i}-T_{1i})}{R}(\alpha_1\alpha_2)^{1/2}\int_0^\infty du\frac{\exp(-u^2t/\tau_1)(u\cos u - \sin u)^2}{[\alpha_1(k_2^2u^2\sin^2 u) + \alpha_2\{k_1(u\cos u - \sin u) + k_2\sin u\}^2]}. \quad (15)$$

For $t \gg \tau$ one has

$$T_1 = T_2 = T_{2i} + \frac{1}{6(\pi)^{1/2}}(T_{1i}-T_{2i})\frac{k_1(\alpha_1)^{1/2}}{k_2(\alpha_2)^{1/2}}\left(\frac{\tau_1}{t}\right)^{3/2}. \quad (16)$$

Thus everything eventually becomes the initial temperature of the medium. One also has for $t \gg \tau$

$$\frac{q(t)}{4\pi R^2} = -\frac{1}{12(\pi)^{1/2}}k_1\frac{k_1(\alpha_1)^{1/2}}{k_2(\alpha_2)^{1/2}}\frac{(T_{2i}-T_{1i})}{R}\left(\frac{\tau_1}{t}\right)^{5/2}. \quad (17)$$

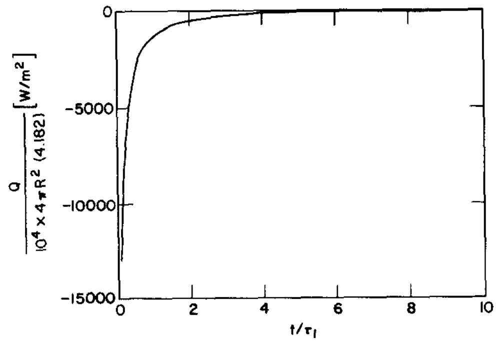

FIG. 1. Heat flux $\dfrac{q}{4\pi R^2(4.182\times 10^4)}$ [W/m²] vs t/τ_1.

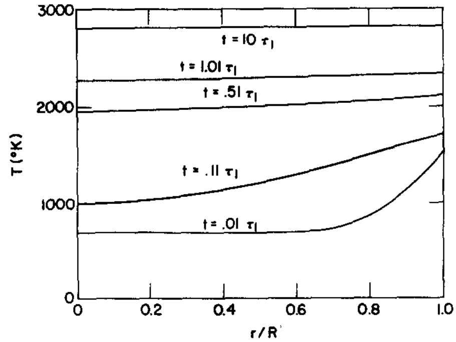

FIG. 2. Temperature profile inside sphere of sodium initially at 700K.

The short time behavior of the solution is obtained from the inverse Laplace transform of equation 6(a). Thus for $t \ll \tau_1$ we have

$$T_1 = T_{1i}+\frac{k_2 R}{\Sigma r}(T_{2i}-T_{1i})\Big\{(\tau_2)^{1/2}\Big[\operatorname{erfc}\Big(\frac{R-r}{2(\alpha_1 t)^{1/2}}\Big)-\operatorname{erfc}\Big(\frac{R+r}{2(\alpha_1 t)^{1/2}}\Big)\Big]$$
$$+\frac{k_1[(\tau_1)^{1/2}+(\tau_2)^{1/2}]}{\Sigma\beta}\Big[\operatorname{erfc}\Big(\frac{R-r}{2(\alpha_1 t)^{1/2}}\Big)-\exp\Big[\frac{\beta(R-r)}{(\alpha_1)^{1/2}}+\beta^2 t\Big]\operatorname{erfc}\Big(\frac{R-r}{2(\alpha_1 t)^{1/2}}+\beta(t)^{1/2}\Big)$$
$$-\operatorname{erfc}\Big(\frac{R+r}{2(\alpha_1 t)^{1/2}}\Big)+\exp\Big[\frac{\beta(R+r)}{(\alpha_1)^{1/2}}+\beta^2 t\Big]\operatorname{erfc}\Big(\frac{R+r}{2(\alpha_1 t)^{1/2}}+\beta(t)^{1/2}\Big)\Big]\Big\} \quad (18)$$

$$T_2 = T_{2i}+\frac{k_1 R}{\Sigma r}(T_{1i}-T_{2i})\Big\{(\tau_1)^{1/2}\operatorname{erfc}\Big(\frac{r-R}{2(\alpha_2 t)^{1/2}}\Big)$$
$$-\frac{k_2[(\tau_1)^{1/2}+(\tau_2)^{1/2}]}{\Sigma\beta}\Big[\operatorname{erfc}\Big(\frac{r-R}{2(\alpha_2 t)^{1/2}}\Big)-\exp\Big[\frac{\beta(r-R)}{(\alpha_2)^{1/2}}+\beta^2 t\Big]\operatorname{erfc}\Big(\frac{r-R}{2(\alpha_2 t)^{1/2}}+\beta(t)^{1/2}\Big)\Big]\Big\}. \quad (19)$$

where $\beta = (k_2-k_1)/\Sigma$, $\Sigma = k_1(\tau_1)^{1/2}+k_2(\tau_2)^{1/2}$.

At $r = R$ the leading short time behavior is given by

$$T_{if}(t) = \frac{k_1(\tau_1)^{1/2}T_{1i}+k_2(\tau_2)^{1/2}T_{2i}}{\Sigma}+\frac{2k_1k_2[(\tau_1)^{1/2}+(\tau_2)^{1/2}]}{\Sigma^2}(t/\pi)^{1/2}(T_{2i}-T_{1i}). \quad (20)$$

4. LIQUID SODIUM INSIDE URANIUM DIOXIDE

In this section we examine the temperature profile inside the sphere as a function of $\tau_1 = R^2/\alpha_1$ as well as considering the heat flux at the interface. We choose the case of liquid sodium at 700K entrained in UO_2 at 3000K since that case is interesting in hypothetical fast breeder reactor postulated accidents, and it is interesting to compare these results with those of the plane interface case [1]. Using the thermophysical properties stated in [1] and a sphere radius of 10^{-4} m, we find the heat flux as shown in Fig. 1. We notice that by $10\tau_1$ the heat flux becomes insignificant. The approximate expression for T_{if}, equation (20), was found to be 5% low at $0.3\tau_1$, 10% low at τ_1, and stayed within 10% of the exact result up to $5\tau_1$. The large time behavior of T_{if}, equation (16), was extremely accurate (1%) after $t = 50\tau_1$.

In Fig. 2 we show the temperature profile in the sphere at selected times between $t = 0.01\tau_1$ and $t = 10\tau_1$, as obtained from equation 34.

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